Dynamics of Tangent

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Abstract

We discuss the dynamics of the family of meromorphic maps $z \mapsto \lambda \tan z$. The Julia sets of these maps have several interesting properties which are strikingly different from those of rational maps. For example, the Julia set is a smooth submanifold of the plane for every $\lambda > 1$; for families of rational maps, those whose Julia sets are smooth submanifolds are isolated points.

Julia sets for complex analytic maps are often "fractals" in the sense of [Ma]. There are certain exceptions. For example, the Julia set of $z \mapsto z^2$ is the unit circle and the Julia set of $z \mapsto z^2 - 2$ is the closed interval $[-2, 2]$. Within the well studied quadratic family $z \mapsto z^2 + c$, these are essentially the only exceptions. Julia sets for rational maps are also often Cantor sets in the plane; examples are the quadratic maps with large $|c|$. They may also be the entire plane; this occurs for the Lattés example. No examples of entire transcendental functions whose Julia sets are Cantor sets are known. In the examples which have been studied (the exponential and the sine families) the Julia sets are either the entire plane or form "Cantor bouquets" [DT]. Here we describe the Julia sets of the meromorphic (non-rational) family $z \mapsto \lambda \tan z$. For $\lambda > 1$ the Julia set is the real line and for $|\lambda| < 1$ it is a

Cantor set. In this note we prove these facts and describe other dynamical phenomena connected with maps in this family.

One of the principal differences arising in the iteration of meromorphic (non-rational) functions is the fact that, strictly speaking, iteration of these maps does not lead to a dynamical system. Infinity is an essential singularity for such a map, and so the map cannot be extended continuously to infinity. Hence the forward orbit of any pole terminates, and, moreover, any preimage of a pole also has a finite orbit. All other points have well defined forward orbits.

Despite the fact that certain orbits of a meromorphic map are finite, the iteration of such maps is important. For example, the iterative processes associated to Newton's method applied to entire functions often yields a meromorphic function as the root-finder. See [CGS]

Let

$$T_\lambda(z) = \lambda \tan(z) = \frac{\lambda}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}.$$

To define the Julia set of this map, we first recall that a point z is called *stable* if there is a neighborhood U of z such that the iterates T_λ^n are uniformly bounded on U. The metric here is the standard Euclidean metric. The Julia set, denoted $J(T_\lambda)$, is the complement of the set of stable points. It is known that $J(T_\lambda)$ has three different equivalent formulations:

1. $J(T_\lambda)$ is the closure of the set of repelling periodic points.
2. $J(T_\lambda)$ is the set of points at which the family of iterates T_λ^n fails to be a normal family in the sense of Montel.
3. $J(T_\lambda)$ is the closure of the set which consists of the union of all of the preimages of the poles of T_λ.

We refer to [DK] for details.

Proposition 1. *If $\lambda \in \mathbf{R}$, $\lambda > 1$, then $J(T_\lambda)$ is the real line and all other points tend asymptotically to one of two fixed sinks located on the imaginary axis.*

Proof. Write $T_\lambda(z) = L_\lambda \circ E(z)$ where

$$\begin{aligned} E(z) &= \exp(2iz) \\ L_\lambda(z) &= -\lambda i\left(\frac{z-1}{z+1}\right) \end{aligned}.$$

E maps the upper half plane onto the unit disk minus 0 and L_λ maps the disk back to the upper half plane. Both E and L_λ preserve boundaries, so T_λ

maps the interior of the upper half plane into itself. Now T_λ also preserves the imaginary axis and we have

$$T_\lambda(iy) = i\lambda \tanh(y).$$

The graph of $\lambda \tanh y$ shows that T_λ has a pair of attracting fixed points located symmetrically about 0 if $\lambda > 1$. See Fig. 1. By the Schwarz Lemma, all points in the upper (resp. lower) half-plane tend under iteration to one of these points.

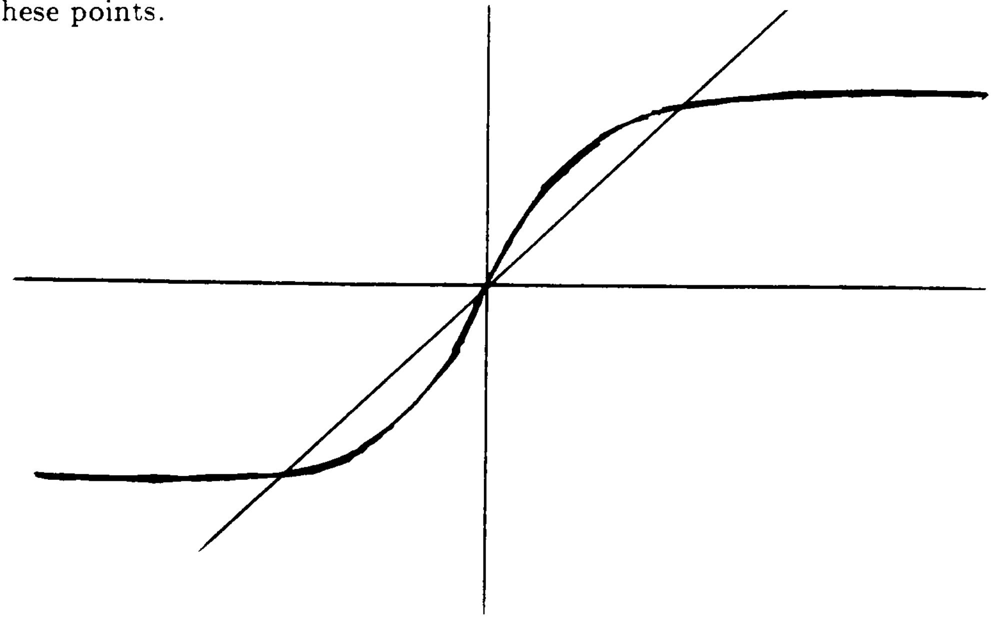

Fig. 1. The graph of $\lambda \tanh x$ when $\lambda > 1$.

Hence neither the upper nor the lower half plane is in $J(T_\lambda)$. The real line is in $J(T_\lambda)$. This follows from the facts that the real line satisfies $T_\lambda^{-1}(\mathbf{R}) \subset \mathbf{R}$ and $T_\lambda(\mathbf{R}) = \mathbf{R} \cup \infty$, and that $T_\lambda'(x) > 1$ for all $x \in \mathbf{R}$ if $\lambda > 1$ ($T_\lambda'(x) \geq 1$ if $\lambda = 1$). Each interval of the form

$$\left(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi\right)$$

is expanded over all of $\mathbf{R}$. If U is any open interval in $\mathbf{R}$, then there is an integer k such that $T_\lambda^k(U)$ covers one of these intervals of length π. Hence $T_\lambda^{k+1}(U)$ covers U. It follows that there exists repelling fixed points and poles of T_λ^{k+1} in U.

q.e.d.

Remarks.

1. If $\lambda = 1$, the $J(T_\lambda) = \mathbf{R}$, and all points with non-zero imaginary parts tend asymptotically to the neutral fixed point at 0.
2. When $\lambda < -1$, the dynamics of T_λ are similar to those for $\lambda > 1$, except that T_λ has an attracting periodic point of period two which hops between the upper and lower half-plane. Since $|T'_\lambda(x)| > 1$ for $x \in \mathbf{R}$, it follows as above that $J(T_\lambda) = \mathbf{R}$ for $\lambda < -1$.

For $0 < |\lambda| < 1$, 0 is an attracting fixed point for T_λ. In this case, the Julia set of T_λ breaks up into a Cantor set, as we show below in Proposition 2. We will employ symbolic dynamics to describe the Julia set in this case. Let Γ denote the set of one-sided sequences whose entries are either integers or the symbol ∞. If ∞ is an entry in a sequence, then we terminate the sequence at this entry, i.e., Γ consists of all infinite sequences $(s_0, s_1, s_2, \ldots)$ where $s_j \in \mathbf{Z}$ and all finite sequences of the form $(s_0, s_1, \ldots, s_j, \infty)$ where $s_i \in \mathbf{Z}$.

The topology on Γ was described in [Mo]. For completeness, we will recall this topology here. If $(s_0, s_1, s_2, \ldots)$ is an infinite sequence, we choose as a neighborhood basis of this sequence the sets

$$U_k = \{(t_0, t_1, \ldots) \mid t_i = s_i \text{ for } i \leq k\}.$$

If, on the other hand, the sequence is finite $(s_0, \ldots, s_j, \infty)$, then we choose the U_k as above for $k \leq j$ as well as sets of the form

$$V_\ell = \{(t_0, t_1, \ldots) \mid t_i = s_i \text{ for } i \leq j \text{ and } |t_{j+1}| \geq \ell\}$$

for a neighborhood basis.

There is a natural map $\sigma : \Gamma \to \Gamma$ called the shift automorphism which is defined by $\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 \ldots)$. Note that $\sigma(\infty)$ is not defined. In Moser's topology, σ is continuous and Γ is a Cantor set. The pair Γ, σ is often called the shift on infinitely many symbols. Γ provides a model for many of the Julia sets of maps in our class, and $\sigma \mid \Gamma$ is conjugate to the action of F on $J(F)$. One such instance of this is shown in the following proposition.

Proposition 2. *Suppose $\lambda \in \mathbf{R}$ and $0 < |\lambda| < 1$. Then $J(T_\lambda)$ is a Cantor set in $\hat{\mathbf{C}}$ and $T_\lambda | J(T_\lambda)$ is topologically conjugate to $\sigma|\Gamma$.*

Proof. Since $0 < |\lambda| < 1$, 0 is an attracting fixed point for T_λ. Let B denote the component of the basin of attraction of 0 in $\mathbf{R}$. B is an open interval of the form $(-p, p)$ where $T_\lambda(\pm p) = \pm p$. (The points $\pm p$ lie on a periodic orbit of period two if $-1 < \lambda < 0$.) The preimages $T_\lambda^{-1}(B)$ consist of infinitely many disjoint open intervals. Let I_j, $j \in \mathbf{Z}$, denote the complementary intervals, enumerated left to right so that I_0 abuts p. See Fig. 2. Then $T_\lambda : I_j \to (\mathbf{R} \cup \infty) - B$ for each j, and $|T'_\lambda(x)| > 1$ for each $x \in I_j$. Standard arguments [Mo] then show that

$$\Lambda = \{x \in \mathbf{R} \cup \{\infty\} \mid T^j_\lambda(x) \in \cup I_j \text{ for all } j\}$$

is a Cantor set and $T_\lambda|\Lambda$ is conjugate to $\sigma|\Gamma$.

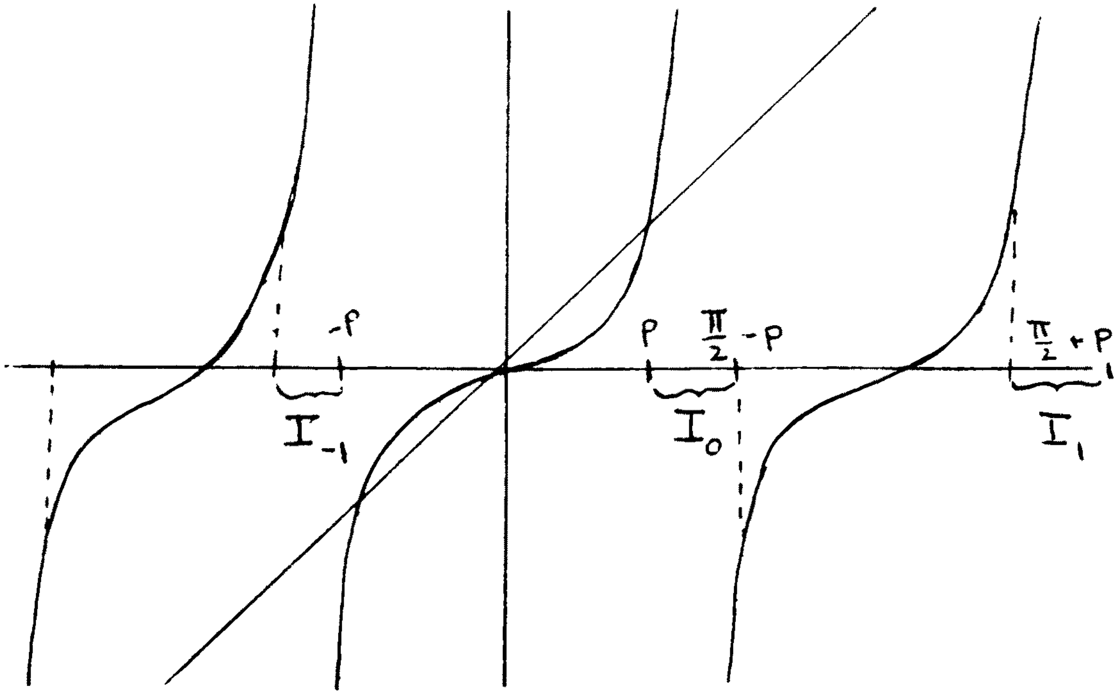

Fig. 2. The intervals I_j.

Now Λ is invariant under all branches of the inverse of T_λ. It therefore contains preimages of poles of all orders and is closed. Hence Λ is the Julia set of T_λ. The classification of stable regions tells us that all other points lie in the basin of 0.

q.e.d.

Remark. The basin of 0 is therefore infinitely connected. This contrasts with the situation for polynomial or entire maps in which *finite* attracting fixed points always have a simply connected immediate basin of attraction.

The Julia set of T_λ is a similar Cantor set for all λ with $|\lambda| < 1$, as shown in the following Proposition.

Proposition 3. *Suppose $0 < |\lambda| < 1$. Then $J(T_\lambda)$ is a Cantor set and each T_λ is quasi-conformally conjugate to $T_{1/2}$.*

Proof: We use techniques of Teichmüller space theory for the proof. The origin is the only attracting fixed point of $T_{1/2}$. There is a holomorphic a map ϕ from the unit disk U to a neighborhood of the origin so that

$$\psi = \phi \circ G \circ \phi^{-1} \text{ has the form } \varsigma \to 1/2\varsigma.$$

Therefore we can find an annulus $A = \{\varsigma \mid r/2 < |\varsigma| < r\}$ which is mapped by ψ onto the annulus $A' = \{\varsigma(r/2)^2 < |\varsigma| < r/2\}$.

Define a Beltrami differential (a measurable structure) on A; that is, let $\hat{\mu}(\varsigma)$ be an arbitrary measurable function on A such that ess sup $|\hat{\mu}(\varsigma)| < 1$. Extend it to U using the map ψ. Pull this Beltrami differential back to the neighborhood of 0 by

$$\mu(z) = \hat{\mu} \circ \psi^{-1}.$$

Extend $\mu(z)$ to the rest of the orbit of $\psi^{-1}(A)$ as follows:

$$\mu(z) = \mu(T^n_{1/2}(z))\overline{(T^n_{1/2})'}(z)/(T^n_{1/2})'(z), \text{ for } z \in T^{-n}_{1/2}(z).$$

Note that this defines the structure everywhere in the stable set. Set $\mu(z) \equiv 0$ everywhere else in $\mathbf{C}$. Use the measurable Riemann mapping theorem to find a quasiconformal map g of the plane whose Beltrami differential is μ. g is uniquely determined by what it does to two points. We will assume that g fixes the origin and maps the asymptotic values to a pair of points symmetric with respect to the origin. The map $F = g^{-1} \circ T_{1/2} \circ g$ is meromorphic by construction. It has exactly two finite asymptotic values and no critical points so by Nevanlinna's Theorem [N], since it fixes 0 and its asymptotic values are symmetric, it is of the form T_λ for some λ. Since g is quasiconformal, it conjugates stable points to stable points. The Julia set is a quasiconformal image of the Julia set of $T_{1/2}$ hence is a Cantor set.

q.e.d.

Remarks.

1. It would be interesting to describe the complete bifurcation diagram for this family.
2. The dynamics of meromorphic (non-rational) maps differs from entire functions or rational maps in other ways. See [DK].

References

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